Fuzzy Modelling of Nonlinear Systems for Stability Analysis Based on Piecewise Quadratic Lyapunov Functions

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***Abstract*—This paper presents a constructive Takagi-Sugeno fuzzy modeling method for a general class of nonlinear systems. This method is particularly suitable for stability analysis based on piecewise quadratic Lyapunov functions. The modeling error is appropriately inserted into the model and an algorithm is proposed to automatically determine the model parameters to keep the modeling error smaller than a desired upper bound. Based on the constructed fuzzy model, exponential stability analysis is performed and the stability constraints are transformed into linear matrix inequalities. Modeling error is also included in the stability analysis to validate the results for the original nonlinear system. The way to utilize the modeling method and stability analysis to systematically find a Lyapunov function for a nonlinear system is demonstrated via an example and the potential capability of the method in estimating the domain of attraction is discussed.**

I. Introduction

STABILITY analysis of nonlinear systems, due to the behavioral variety of these systems and theoretical complicacy of the analysis methods, is still a challenging area in the control systems engineering. Direct and indirect Lyapunov stability theorems, [1], are supposed to be the only powerful analytical tools for stability analysis of the equilibria of a nonlinear system. Although the indirect Lyapunov theorem, together with the centre manifold theorem, can establish the stability/instability of the nonlinear system based on the stability/instability of its linearized version, this analysis is local and unable to investigate global stability or give an estimate of the Domain of Attraction (DOA). In order to analyze global stability or analytically estimate the DOA, we need to use the direct Lyapunov stability and consequently find a Lyapunov function. However, finding a Lyapunov function for a given nonlinear system is usually a very difficult task and is actually the main problem in stability analysis of a nonlinear system.

So far, lots of attempts have been made to develop a systematic method to find a Lyapunov function for a general class of nonlinear systems. Stability analysis based on Takagi-Sugeno (TS) fuzzy model of the nonlinear system is one of the approaches proposed in the early 1990s to achieve this objective, [2]. In this approach of stability analysis, the nonlinear system is first replaced by its TS fuzzy model. Then the stability analysis is performed based on quadratic, piecewise quadratic, or fuzzy Lyapunov functions and the stability constraints are transformed into Linear Matrix Inequalities (LMIs). The Lyapunov function is then automatically obtained by solving the LMIs via currently available powerful software. However, lots of the results obtained so far have disregarded the modeling error between the original nonlinear system and its TS fuzzy model, [3], [4], and consequently suffer from an important theoretical deficiency.

In this paper, we propose a constructive fuzzy modeling method based on triangular Fuzzy Basic Functions (FBFs) which is very suitable for stability analysis using continuous piecewise quadratic Lyapunov functions. Similar to the structure introduced in [5], we insert the modeling error in the TS model as vanishing perturbation and present an algorithm to automatically determine the parameters of the TS model to satisfy a predefined modeling precision. Using the final TS model and previously satisfied bounds on the modeling error, we perform the exponential stability analysis based on continuous piecewise quadratic Lyapunov function and derive the stability constraints in the form of LMIs. These stability constraints are valid for the original nonlinear system, because they properly include the effects of modeling error. Finally, using a clarifying example, we show how to utilize the proposed modeling method and stability analysis to systematically find the Lyapunov function of a relatively complicated nonlinear system and introduce the potential capability of this method in estimating the DOA of an equilibrium point of the nonlinear system.

II. Fuzzy Modelling

Consider the nonlinear system

$$\dot{x} = A(x)x \qquad (1)$$

where $x \in R^n$ is the state vector. Using a TS fuzzy system, in this section we try to systematically approximate (model) the nonlinear system (1) in a way that the approximation (modeling) error satisfies a predefined upper bound.

The lth fuzzy rule in a TS fuzzy system can be written as

$$\begin{aligned} R_l:&\ \text{IF } x_1 \text{ is } H_{l1} \text{ and } x_2 \text{ is } H_{l2} \text{ and} \ldots x_n \text{ is } H_{ln} \\ &\ \text{THEN } \dot{x} = A_l x, \qquad l = 1, 2, \ldots, L \end{aligned} \qquad (2)$$

where H_{lj} is a fuzzy set, L is the number of fuzzy rules, and A_l is the state matrix. Based on some conventional assumptions described in section II.A, the output of the TS fuzzy system can be written as

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$$\dot{x} = \sum_{l=1}^{L} \mu_l(x) A_l x \tag{3}$$

where $\mu_l(x)$ is the weight of the lth subsystem and

$$\sum_{l=1}^{L} \mu_l(x) = 1, \qquad 0 \le \mu_l(x) \le 1, \qquad l = 1,2,\ldots,L \tag{4}$$

In order to describe the modeling method, we first recall the structure of a Multi Input-Single Output (MISO) fuzzy system and the concept of FBFs from [6].

A. MISO Fuzzy Systems and FBFse

Consider a fuzzy system $\varphi : U \mapsto V$, where $U = U_1 \times U_2 \times \cdots \times U_n \subset R^n$ is the input space and $V \subset R$ is the output space. Suppose the fuzzy system consists of fuzzifier, fuzzy rule base, inference engine, and defuzzifier. The fuzzy rule base consists of $L = \prod_{j=1}^{n} L_j$ rules as follows

$$\begin{aligned} R_{i_1 i_2 \cdots i_n} &: \text{IF } x_1 \text{ is } H_{i_1}^1 \text{ and } x_2 \text{ is } H_{i_2}^2 \text{ and} \ldots x_n \text{ is } H_{i_n}^n \\ &\quad \text{THEN } y \text{ is } C_{i_1 i_2 \cdots i_n} \\ &\quad i_1 \in \{1,\ldots,L_1\}, i_2 \in \{1,\ldots,L_2\}, \ldots, i_n \in \{1,\ldots,L_n\} \end{aligned} \tag{5}$$

where $x_j \, (j = 1,2,\ldots,n)$ are input variables and y is the output. The fuzzy sets $H_{i_j}^j \subset U_j$ and $C_{i_1 i_2 \cdots i_n} \subset V$ are linguistic terms which are described by the membership functions $H_{i_j}^j(x_j)$ and $C_{i_1 i_2 \cdots i_n}(y)$, respectively.

According to [6] and under the four main assumptions that: the fuzzifier is a singleton fuzzifier, the T-norm in fuzzy implication and inference is algebraic product, the defuzzifier is a centre of average defuzzifier, and $C_{i_1 i_2 \cdots i_n}$ is a normal fuzzy set, the output of the fuzzy system is formulized as

$$y = \varphi(x) = \sum_{i_1 i_2 \cdots i_n \in I} Q_{i_1 i_2 \cdots i_n}(x) y_{i_1 i_2 \cdots i_n} \tag{6}$$

where $I = \{ i_1 i_2 \cdots i_n \mid i_j = 1,2,\ldots,L_j; \, j = 1,2,\ldots,n \}$ includes all indexes of fuzzy rules and $Q_{i_1 i_2 \cdots i_n}(x), \, (i_1 i_2 \cdots i_n \in I)$ are FBFs which are decomposed to the FBFs $Q_{i_j}^j(x_j)$ as

$$Q_{i_1 i_2 \cdots i_n}(x) = \prod_{j=1}^{n} Q_{i_j}^j(x_j), \qquad \forall i_1 i_2 \cdots i_n \in I \tag{7}$$

$$Q_{i_j}^j(x_j) = \frac{H_{i_j}^j(x_j)}{\sum_{i_j=1}^{L_j} H_{i_j}^j(x_j)}, \quad i_j = 1,2,\ldots,L_j; \quad j = 1,2,\ldots,n \tag{8}$$

In this paper, we choose the membership functions $H_{i_j}^j(x_j)$ to be triangular functions and parameterize them such that

$$\sum_{i_j=1}^{L_j} H_{i_j}^j = 1, \qquad j = 1,2,\ldots,n$$

Therefore the membership functions take the shape shown in Fig. 1, and according to (8) they are equal to FBFs $Q_{i_j}^j(x_j)$, i.e. $Q_{i_j}^j(x_j) = H_{i_j}^j(x_j), \, j = 1,2,\ldots,n$.

These triangular FBFs partition the input (state) space into hypercubic cells S_j, as shown in Fig. 2 for 2-dimensional space. It can be seen in section III that this partitioning property is very useful for stability analysis based on piecewise quadratic Lyapunov functions.

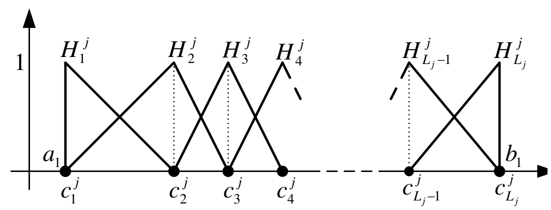

Fig. 1. Triangular membership functions related to the input (state) variable x_j.

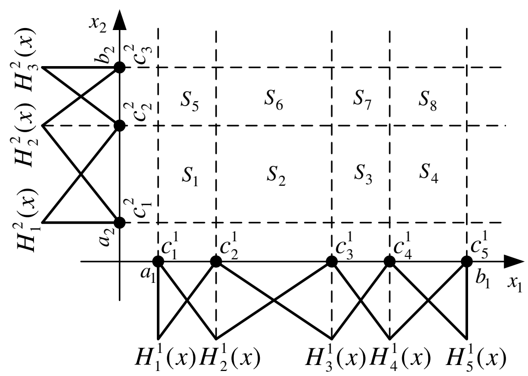

Fig. 2. Partitioning the state space by triangular FBFs.

B. TS Fuzzy Modeling Based on Triangular FBFs

Consider the TS fuzzy system (3) with the fuzzy rules (2). We can rewrite the fuzzy rules (2) as

$$\begin{aligned} R_l &: \text{IF } x_1 \text{ is } H_{i_1}^1 \text{ and } x_2 \text{ is } H_{i_2}^2 \text{ and} \ldots x_n \text{ is } H_{i_n}^n \\ &\quad \text{THEN } \dot{x} = A_l x + B_l u \qquad l \equiv i_1 i_2 \cdots i_n \\ &\quad i_1 \in \{1,\ldots,L_1\}, i_2 \in \{1,\ldots,L_2\}, \ldots, i_n \in \{1,\ldots,L_n\} \end{aligned} \tag{9}$$

Comparing (9) with the MISO system rules (5), we see that each R_l in (9) or (2) corresponds to one rule $R_{i_1 i_2 \cdots i_n}$ in (5). The only deference is that the consequent of the TS fuzzy rules is a dynamical system, rather than a single variable in the MISO fuzzy rules (5). In this section we show how to use the MISO system (6) to obtain the TS fuzzy approximation (3) of the nonlinear system (1). The main idea is to separately approximate each entry of the matrix

$A(x)$ with a single MISO fuzzy system, but with the same FBFs for all entries. Choosing the same FBFs for all entries, makes it possible to combine all separate approximation into a compact form and obtain the TS fuzzy model. The weight $\mu_l(x)$ of each subsystem in (3) is then equal to an FBF as following

$$\mu_l(x) \equiv Q_{i_1 i_2 \cdots i_n}(x), \qquad \forall i_1 i_2 \cdots i_n \in I; \qquad l \equiv i_1 i_2 \cdots i_n \tag{10}$$

and according to (7) and (8), the property (4) will also hold for the weights $\mu_l(x)$. In the sequel we describe further details of implementing the abovementioned idea.

As we showed in section 2.1, the triangular FBFs partition the state space in a domain $D = [a_1, b_1] \times \cdots \times [a_n, b_n] \ni 0_{n\times 1}$ into cells S_j. Each vertex of a cell is the centre of one of the neighbour FBFs. In a cell S_j, we can represent the nonlinear system (1) as

$$\dot{x} = A(x)x = \sum_{i\in I(j)} \mu_i(x) A_i x + \delta_A^j(x), \quad x \in S_j, \;\; j \in J \tag{11}$$

where $\delta_A^j(x)$ is a perturbation term containing the modelling error, and $I(j)$ is the set of all indexes of subsystems which are active in the cell S_j. J is a set containing the indexes of all cells. The error term is calculated as following

$$\delta_A^j(x) = \Delta_A^j(x)x, \;\; \Delta_A^j(x) = A(x) - \sum_{i\in I(j)} \mu_i(x) A_i \tag{12}$$

The main objective of the modelling procedure is to determine the model parameters, i.e. matrices A_l and centres and number of membership functions, to satisfy the following precision constraint in all cells

$$\sup_{x\in S_j} \left| a_{pq}(x) - \sum_{i\in I(j)} \mu_i(x) a_{pq}^i \right| \le \varepsilon_{pq}^A, \qquad j \in J \tag{13}$$

where $a_{pq}(x)$, a_{pq}^i, ε_{pq}^A are the pqth entry of the matrices $A(x)$, A, and E_A, respectively. E_A is the predefined desired precision matrix. We can write (13) in the following more compact form

$$\left|\Delta_A^j(x)\right| \preceq \mathrm{E}_A, \qquad x \in S_j, \quad j \in J \tag{14}$$

where $\left|\Delta_A^j(x)\right|$ is a matrix containing the absolute value of the entries of matrix $\Delta_A^j(x)$ and the matrix inequality $Y \preceq Z$ means that each entry of Y is smaller than or equal to its corresponding entry in Z.

We choose $A_i = A(c_i)$, where c_i is the centre of the weight $\mu_i(x)$. So, the TS fuzzy model is exact at the centre of weights, i.e. $\delta_A^j(c_i) = 0, \, i \in I(j), \, j \in J$. Based on this way of evaluating matrices A_i, we propose an algorithm to automatically determine the centre of membership functions to satisfy the precision constraint (14). Before describing the algorithm, we need to first define the following vectors

$$\begin{aligned} g(x) &= \left[a_{11}(x), \ldots, a_{1n}(x), a_{21}(x), \ldots, a_{2n}(x), \ldots \right. \\ &\qquad \left. , a_{n1}(x), \ldots, a_{nn}(x)\right]^T \\ \phi_j(x) &= \sum_{i\in I(j)} \mu_i(x) \left[a_{11}^i, \ldots, a_{1n}^i, a_{21}^i, \ldots, a_{2n}^i, \ldots \right. \\ &\qquad \left. , a_{n1}^i, \ldots, a_{nn}^i\right]^T \\ \varepsilon &= \left[\varepsilon_{11}^A, \ldots, \varepsilon_{1n}^A, \varepsilon_{21}^A, \ldots, \varepsilon_{2n}^A, \ldots, \varepsilon_{n1}^A, \ldots, \varepsilon_{nn}^A\right]^T \\ \bar{e}_j(x) &= \frac{g(x) - \phi_j(x)}{\varepsilon} \end{aligned} \tag{15}$$

The vector $g(x)$ contains all the entries of matrix $A(x)$. Each element of this vector is a scalar nonlinear MISO function which we are going to approximate by a MISO fuzzy system (6). Each element of the vector $\varphi_j(x)$ is the fuzzy approximate of its corresponding element in vector $g(x)$ inside the cell S_j. The vector $\bar{e}_j(x)$ is the normalized modelling error in the cell S_j and the vector division operator in (15) means an element by element division. The precision constraint (14) is then written as

$$\sup_{x\in S_j} \left|\bar{e}_j(x)\right| \preceq 1, \qquad \forall j \in J \tag{16}$$

where $\left|\bar{e}_j(x)\right|$ is a vector containing the absolute value of the elements of vector $\bar{e}_j(x)$.

Algorithm: Given the precision matrix E_A and the maximum number of cells *N*:

Step 1: Initialize the centre of membership functions to partition the state space in domain *D* into quadrants. In other words, divide each dimension of state space into two intervals $[a_i, 0]$ and $[0, b_i]$. The initial structure of membership functions and partitioning of domain *D* is shown in Fig. 3. This initial partitioning ensures that the centre of one of the weights $\mu_i(x)$ is placed on origin, and so the model is exact at origin.

Step 2: Compute the normalized error vector for all existing cells based on (15). If the constraint (16) is satisfied for all cells, terminate the algorithm, otherwise store in an array $\hat{J}$ the indexes of all cells in which the constraint (16) is not satisfied. For example, supposing that in Fig. 4 the constraint (16) is not satisfied in the shaded cells, we have $\hat{J} = \{3, 4\}$.

Step 3: In each dimension, bisect the partitioning intervals whose bisection causes bisecting the cells $S_j, j \in \hat{J}$. For example, in the first dimension (x_1) in Fig. 4 we need to bisect both intervals $[a_1, 0]$ and $[0, b_1]$, which is shown by dotted lines. In other hand, in the second dimension (x_2) we need to only bisect the interval $[0, b_2]$ which is shown by dashed line in Fig. 4.

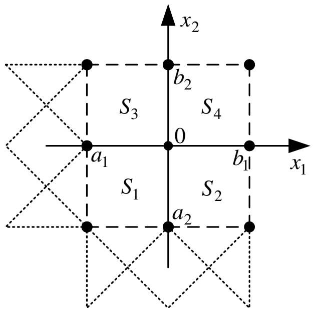

Fig. 3. Initial partitioning of modeling domain D by initial membership functions.

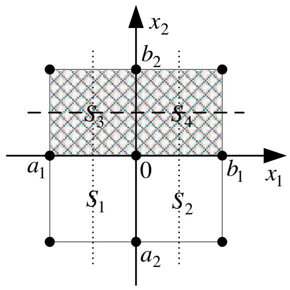

Fig. 4. Initial partitioning of modeling domain D by initial membership functions.

Step 4: For each dimension, supposing that the intervals are bisected through the way described in step 3, compute the normalized error of the new half-cells, $\overline{e}_{j1}$ and $\overline{e}_{j2}$, and store them in a vector array as

$$\overline{\mathbf{e}}_i = \left\{\overline{e}^i_{j1}, \overline{e}^i_{j2}\right\}, \quad j \in \hat{J}, \qquad i = 1,2,\ldots,n$$

Note that there is equal number of vectors in the array $\overline{\mathbf{e}}_i$ of all dimensions.

Step 5: For each dimension, compute the number of all added cells v_i which are generated in the bisecting procedure in step 4, e.g. in Fig. 4 we have $v_1 = 4$ and $v_2 = 2$.

Step 6: Based on the following criteria, choose the bisections in one dimension, construct the new cells, and compute the new centres of membership functions and matrices A_i:

1) Check if all the vectors in the array $\overline{\mathbf{e}}$ of each dimension satisfy the constraint (16). If (16) is satisfied for the array $\overline{\mathbf{e}}$ of only one dimension, choose the bisections in that dimension and proceed to step 7, else if it is satisfied in several dimensions, keep the array $\overline{\mathbf{e}}$ of those dimensions and drop the array of others, i.e. drop other dimensions from the existing options.
2) Among the remaining dimensions, If only one dimension has the smallest v_i, choose the bisections in that dimension and proceed to step 7, else if several dimensions have the smallest v_i, keep the array $\overline{\mathbf{e}}$ of those dimensions and drop the array of others.
3) Among the remaining dimensions, choose the bisections in the dimension which has the smallest some of square errors as

$$\mathrm{SSE}_i = \sum_{j \in \hat{J}} \left(\left\|\overline{e}^i_{j1}\right\|_2^2 + \left\|\overline{e}^i_{j2}\right\|_2^2 \right)$$

Step 7: If the number of existing cells is smaller than or equal to *N*, go back to step 2, otherwise terminate the algorithm with the note that using the proposed algorithm, it is impossible to satisfy the constraint (16) in all cells with fewer number of cells than *N*.

The above algorithm automatically determines the parameters of the fuzzy model (11) to satisfy the modelling precision (14). It should be noted that it is very likely that at the end of the algorithm, the suprimum of modelling error in some cells is much smaller than the desired upper bound. So it is recommended to compute the suprimum error matrix E^j_A of each cell at the end of the algorithm and use it in the consequent stability analysis procedure, instead of using the same precision matrix E_A for all cells. Therefore, the modelling error terms in (11) finally satisfies the following precision constraint in each cell

$$\left|\Delta^j_A(x)\right| \preceq \mathrm{E}^j_A, \qquad j \in J \tag{17}$$

It can be easily seen that if (17) holds, the following inequalities also hold with Frobenius, infinity, and 1 norms

$$\left\|\Delta^j_A(x)\right\|_p \le \left\|\mathrm{E}^j_A\right\|_p, \qquad p = F, \infty, 1$$

According to norm equivalencies in finite dimensional spaces, e.g. found in [7], we can easily write the following inequality for the modelling error terms

$$\left(\delta^j_A(x)\right)^T \delta^j_A(x) \le \left\|\Delta^j_A(x)\right\|_2^2 \|x\|_2^2 \le \alpha_j^2 \|x\|_2^2$$
$$,\alpha_j = \min\left\{ \left\|\mathrm{E}^j_A\right\|_F, \sqrt{n}\left\|\mathrm{E}^j_A\right\|_\infty, \sqrt{n}\left\|\mathrm{E}^j_A\right\|_1 \right\} \tag{18}$$

The inequality (18) is used in stability analysis to enter the effect of modelling error into the stability constraints.

III. Stability Analysis

We can rewrite the fuzzy model (11) as

$$\dot{\overline{x}} = \sum_{i \in I(j)} \mu_i(x) \overline{A}_i \overline{x} + \delta^i_A(\overline{x}) u, \qquad x \in S_j; \quad j \in J \tag{19}$$

Where

$$\overline{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \overline{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta^j_A(\overline{x}) = \begin{bmatrix} \delta^j_A(x) \\ 0 \end{bmatrix} \tag{20}$$

Consider the following piecewise quadratic candidate of

Lyapunov function

$$V(x)=\begin{cases} x^T P_j x, & x\in S_j, \quad j\in J_0 \\ \overline{x}^T \overline{P}_j \overline{x}, & x\in S_j, \quad j\in J_1 \end{cases} \tag{21}$$

where $J_0 \subseteq J$ is the set of indexes of cells which contain the origin and $J_1 \subset J$ is the set of indexes of other cells. The symmetric matrixes P_j and $\overline{P}_j$ are parameterized as follows to preserve the continuity of the Lyapunov function (21) at the boundary of cells, [8]

$$\begin{cases} P_j = F_j^T T F_j, & j\in J_0 \\ \overline{P}_j = \overline{F}_j^T T \overline{F}_j, & j\in J_1 \end{cases}$$

where T is an unknown symmetric matrix and the constraint matrices F_j and $\overline{F}_j$ are systematically determined from the information of the fuzzy model via the method described in [8]. The following theorem gives the sufficient conditions for exponential stability of the origin of the nonlinear system (1).

Theorem: The origin of the nonlinear system (1) which is represented by the fuzzy model (11), in which the error terms satisfy the constraint (18), is exponentially stable if a positive constant σ and symmetric matrices T, U_j, and W_j can be found such that all the entries of the matrices U_j and W_j are non-negative and the LMIs

$$P_j - E_j^T U_j E_j > 0 \tag{22}$$

$$\begin{bmatrix} A_i^T P_j + P_j A_i + E_j^T W_j E_j + \sigma\alpha_j^2 I & P_j \\ P_j & -\sigma I \end{bmatrix} < 0 \tag{23}$$

hold for $i\in I(j)$, $j\in J_0$ and the LMIs

$$\overline{P}_j - \overline{E}_j^T U_j \overline{E}_j > 0 \tag{24}$$

$$\begin{bmatrix} \overline{A}_i^T \overline{P}_j + \overline{P}_j \overline{A}_i + \overline{E}_j^T W_j \overline{E}_j + \sigma\alpha_j^2 I & \overline{P}_j \\ \overline{P}_j & -\sigma I \end{bmatrix} < 0 \tag{25}$$

hold for $i\in I(j)$, $j\in J_1$. The constraint matrixes E_j and $\overline{E}_j$ are defined to provide the ability to apply the S-procedure method , [9], and are systematically determined via the method described in [8].

Proof: The proof follows the similar steps as in [8], with some modifications to include the modelling error effects. It can be easily shown (see [8]) that the LMIs (22) and (24) ensure that $c_1\|x\|_2^2 \le V(x) \le c_2\|x\|_2^2$ $c_1>0, c_2>0$, which is the first constraint of exponential stability theorem , [1]. Here we show that the LMIs (23) and (25) imply the satisfaction of the second constraint of exponential stability theorem, i.e. $\dot V(x) \le c_3\|x\|_2^2$, $c_3>0$.

If we compute the derivative of the Lypunov function along the trajectories of the system (19), we have

$$\dot V(x) = \sum_{i\in I(j)} \mu_i(x)\overline{x}^T\left(\overline{A}_i^T\overline{P}_j + \overline{P}_j\overline{A}_i\right)\overline{x} + \delta_j^T(\overline{x})P_j\overline{x} + \overline{x}^T\overline{P}_j\delta_j(\overline{x})$$

It can be easily shown that the following inequality holds with an arbitrary positive constant σ

$$\delta_j^T(\overline{x})P_j\overline{x} + \overline{x}^T\overline{P}_j\delta_j(\overline{x}) \le \frac{1}{\sigma}\overline{x}^T\overline{P}_j^2 x + \sigma\delta_j^T(\overline{x})\delta_j(\overline{x})$$

According to the definition (20) and based on the above inequality and the constraint (18), we can easily write

$$\delta_j^T(\overline{x})P_j\overline{x} + \overline{x}^T\overline{P}_j\delta_j(\overline{x}) \le \frac{1}{\sigma}\overline{x}^T\overline{P}_j^2 x + \sigma\alpha_j^2\|x\|_2^2$$

Therefore we have

$$\dot V(x) \le \sum_{i\in I(j)} \mu_i(x)\overline{x}^T\left(\overline{A}_i^T\overline{P}_j + \overline{P}_j\overline{A}_i + \sigma\alpha_j^2 I + \frac{1}{\sigma}\overline{P}_j^2\right)\overline{x}$$

Now it is sufficient to have

$$\overline{A}_i^T\overline{P}_j + \overline{P}_j\overline{A}_i + \sigma\alpha_j^2 I + \frac{1}{\sigma}\overline{P}_j^2 < 0, \qquad i\in I(j),\ j\in J$$

Adding the S-procedure term $\overline{E}_j^T W_j \overline{E}_j$ to the above inequality (see [8]) and applying the Schur complement, [9], we obtain the LMIs (25). The LMIs (24) are obtained through the same method, so the proof is completed. ■

In the next section we provide an example to clearly show the way to utilize the proposed modelling method and stability analysis to systematically find a Lyapunov function for a general class of nonlinear systems.

IV. Example

Consider the nonlinear system

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} x_1\sin(x_2)-2 & \sin(x_1)\cos(x_2) \\ \cos(x_2) & x_2\sin(x_1)-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{26}$$

Choosing the following precision matrix, we try to model the nonlinear system (26) in the modeling domain $D=\left[-\pi/2,\pi/2\right]\times\left[-\pi/2,\pi/2\right]$

$$\mathrm{E}_A = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

If we apply the algorithm proposed in section 2 to the nonlinear system (26), each dimension of state space is split into four intervals and the state space in domain D is partitioned into 16 cells. As an example, the nonlinear function $a_{12}(x)$, its fuzzy approximate and the modeling error is shown in Fig. 5.

Now based on the obtained fuzzy model, we can easily compute the matrices $\overline{F}_j$, $\overline{E}_j$, and E_A^j for each cell and solve the LMIs (22)–(25). Solving the LMIs with the available powerful toolbox in MATLAB, we systematically obtain the Lyapunov function shown in Fig. 6.

In order to show the potential capability of the presented fuzzy-model-based stability analysis method in estimating the DOA, we draw the contours of the Lyapunov function in the modeling domain D, in which we ensure $\dot V(x)$ is negative definite.

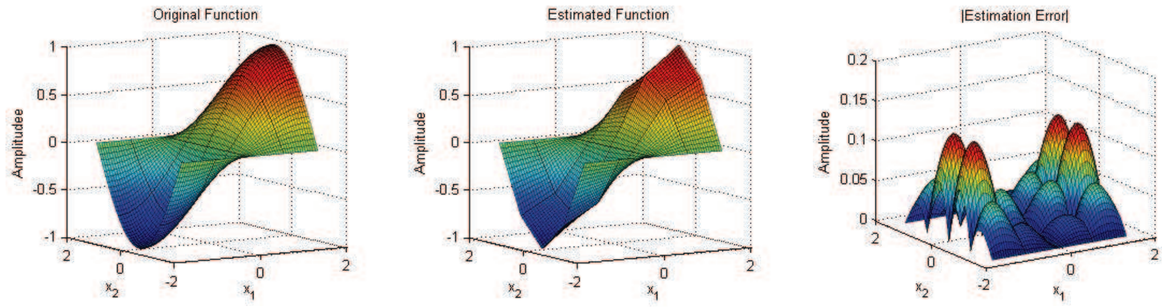

Fig. 5. Piecewise quadratic Lyapunov function of system (26).

As we see in Fig. 7, the Lyapunov function takes the maximum value equal to 7.95 inside the domain *D*. So, according to the method described in (Khalil, 2002) and as shown in Fig. 7, we can propose a conservative estimate of DOA as

$$\Omega = \{ x \in D \mid V(x) \leq 7.95 \}$$

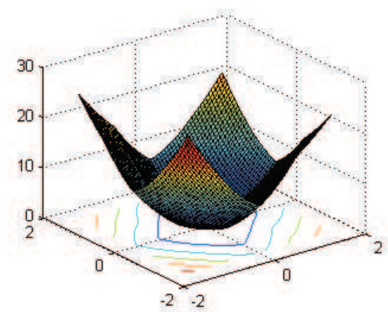

Fig. 6. Piecewise quadratic Lyapunov function of system (26).

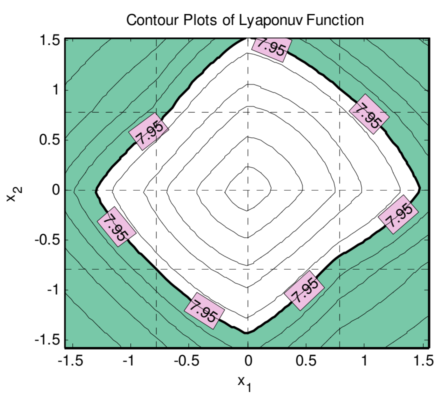

Fig. 7. Contours of the Lyapunov function. The white area shows the estimation of DOA

V. Conclusion

In this paper we presented a constructive method for fuzzy modelling nonlinear systems which is very suitable for systematic stability analysis based on piecewise quadratic Lyapunov functions. We also showed the potential capability of the stability analysis method in estimating the DOA. The authors are currently further developing the proposed method to obtain an efficient and systematic DOA estimation method which is applicable to a general class of nonlinear systems.

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